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A THIRTY-YEAR REFLECTION ON CONSTRUCTIVISM IN MATHEMATICS EDUCATION IN PME

INTRODUCTION

As the International Group for the Psychology of Mathematics Education (IG PME) grew up, so did constructivism. Reflecting over the role of constructivism in the history of mathematics education is a daunting task, but one which provides an opportunity to reflect on what has been accomplished, honor the contributions of scholars around the world, and identify what remains unfinished or unexplained. In undertaking this task, we divide our treatment into five major sections: (1) The historical precedents of constructivism during the first ten years (1976-85); (2) The debates surrounding the ascendancy of constructivism during the next ten years (1986-95); (3) Our own articulation of key principles of constructivism; (4) Thematic developments over the last ten years (1996-present); and (5) An assessment of and projection towards future work. Looking back, we hope we can share the excitement of this epoch period in mathematics education and the contributions to it which came from across the globe.

Since its inception at the 1976 International Congress on Mathematical Education (ICME) in Karlsruhe, PME has addressed three major goals all addressing the need to integrate mathematics education and psychology. While PME clearly has welcomed and thrived on multiple theories of psychology, beginning with Skemp’s (1978) *The Psychology of Learning Mathematics*, it has preferred those with a cognitive, and to some extent, an affective orientation. Two major theories of intellectual development have been dominant, namely constructivism and socio-cultural perspectives. In recent years, these two theories have intermingled, but in this volume, they are separated as we trace their paths, overlapping and distinctive. We will not give in to the frequent temptation to cast constructivism and socio-cultural perspectives as a diametrically opposed where one is personal/individual and the other social; but rather track the evolution of the theory via the theorists and the perspectives that they assign to their work.
PART 1: THE HISTORICAL PRECEDENTS FOR CONSTRUCTIVISM (1976-1985)

We would classify constructivism as a “grand theory” in the typology offered by diSessa and Cobb (2004), in that it was paradigmatic for mathematics education, though as they put it, grand theories are often “too high-level to inform the vast majority of consequential decisions” (p. 80), at least at a level of specificity to guide instructional practice (also see Ernest, 1991b, and Thompson, 2002). To specify practice, constructivism relied on partner instructional theories, such as “Realistic Mathematics Education (RME)” (De Lange, 1987; Freudenthal, 1991; Gravemeijer, 1994), “didactical engineering” (Artigue, 1990; Balacheff, 1990), “cognitively guided instruction” (Carpenter, Fennema, Franke, Levi & Empson, 1999) or “constructionism” (Harel & Papert, 1991), all of which were compatible with the grand theory and were a part of PME deliberations.

As a grand theory, constructivism served as a means of prying mathematics education from its sole identification with the formal structure of mathematics as the sole guide to curricular scope and sequence. It created a means to examine that mathematics from a new perspective, the eyes, mind and hands of the child. Constructivism developed in mathematics education to counter the effects of behaviorism (Gagné, 1965; Thorndike, 1922), which had focused on measurement and the production of patterns and levels of outcomes by stimuli. Constructivism evolved as researchers’ interests in the child’s reasoning went beyond a simple diagnostic view of errors to understanding the richness of student strategy and approach. It took hold in practice, because it addressed the two primary concerns of teachers: (1) Students’ weak conceptual understanding with over-developed procedures (relational vs. instrumental in Skemp’s, 1978, language), and (2) Students demonstrated difficulties with recall and transfer to new tasks. Constructivism did so by focusing the strengths and resources children brought to the tasks, and by making their active involvement and participation central to the theoretical framework.

Understanding how the constructivist movement swept through mathematics education requires one to take an evolutionary look at its inception and development. Some argued that its quick ascension demonstrated the tendency of the field to respond too quickly to fashions (Wheeler, 1987), or even reflected zealotry of the part of its proponents (Kilpatrick, 1987). With time, we can ask what propelled it to such notoriety, looking critically, why was it so often it was the trappings of constructivism, and not the solid conceptual basis that was practiced? Further, it will be helpful to consider if the overall research programme of constructivism is still progressive, static or degenerating in the Lakatosian sense (Lakatos, 1976).

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Lakatos (1976) argued that the best way to describe a paradigm was to identify it as a research programme which consisted of a theoretical hardcore of ideas, core commitments surrounded by a protective belt of theories and finally surrounded by empirical studies. Challenges to the programme such as anomalies would cause changes in the empirical studies first and if necessary to the protective belt of bridging theories. The hardcore could not be directly challenged; although depending on the success of the adjustments, the programme could be cast as progressive (gaining power) or degenerating (losing power).
We locate the roots of constructivism in three traditions, very much a part of the tradition of PME: (1) Problem solving (Garofalo & Lester, 1985; Goldin & Gennain, 1983; Polya, 1957; Schoenfeld, 1985), (2) Misconceptions, critical barriers, and epistemological obstacles (Brousseau, 1983; Confrey, 1990; Driver & Easley, 1978; Hawkins, 1978), and (3) Theories of cognitive development (Krutetskii, 1976; Piaget, 1954; Sinclair, 1987; Van Hiele & Van Hiele-Geldof, 1958). All of these traditions impregnated mathematics education with the view that something more than the logic of mathematics was necessary to explain, predict, and facilitate mathematics learning. They all recognized that the difficulty or ease of learning could not be explained simply by looking at the complexity of the material, but rather that other factors were needed to account for the path learning traversed and levels of success or failure. While behaviorism had presented the simplest account from an external viewer’s perspective (a set of stimulus-response connections), a more complex psychological theory was needed to capture not only the behaviors but the experience of learning. Also, a black box approach (one which lacks constructs to explain non-observable processes) to the mind left one unable to gain explanation, much less prediction, over students’ thinking and reasoning, both alone and within interactions.

The first tradition of problem solving provided a number of key elements for constructivist thought. Polya’s (1957) four stages (understanding, devising a plan, carrying out the plan, and looking back) emphasized that mathematics was more than a set of formal definitions, theorems and proofs and acknowledged the central role of problems in generating new solutions, propelling the field forward. Research on heuristics (Goldin & Gennain, 1983) demonstrated the power of examining the problem solver’s strategies. And the need to be aware of the problem solving process itself promoted increased emphasis on meta-cognition’s role in thinking (Garofalo & Lester, 1985). In problem solving theories, there was always a philosophical sense that the problems existed independent of the solver, and by learning a fruitful set of techniques, solutions could be more easily and effectively be sought. Polanyi (1958), in Personal Knowledge, described it as “looking for it [the solution] as if it were there, pre-existent.” As a result, problem solving was viewed as an acceptable extension to mathematics, not challenging in any fundamental way the epistemological character of the enterprise, but only extending and enhancing it. Debates focused typically only how much time should be devoted to it.

The second major tradition was that of systematic errors and misconceptions. In this, the concept of errors had ripened to include the idea of epistemological obstacles (Bachelard, 1938; Brousseau, 1983; Sierpinska, 1992), misconceptions (Bachelard, 1938; Confrey, 1990), critical barriers (Hawkins, 1978), and alternative conceptions (Driver & Easley, 1978). Typically, practitioners assumed that errors could be eradicated by drawing students’ attention to them and providing them the correct procedures. In contrast, misconceptions seemed to pop back up like weeds, and their attraction to students suggesting some deeper compelling quality. Three examples have been extensively discussed at PME: (1) “Multiplication makes bigger, division makes smaller (MMDMS)” (Greer, 1987);
(2) “The graph as a picture of the path of an object” (Monk & Nemirovsky, 1994); and (3) “Additive equal amounts to numerators and denominators preserves proportionality” (Hart, 1984). Other examples also abound in the literature such as longer numbers are bigger so $1.217 > 1.3$ (Resnick, Nesher, Leonard, Magone, Omanson & Peled, 1989). Explanations for misconceptions required one to find ways to identify cases in which the formation of the generality made sense, and to recognize where an extension of the idea would produce errors. Thus, MMBDMS works for positive rational numbers greater than one, but fails as one extends the meaning of multiplication and division to rational numbers between 0 and 1. In other cases, misconceptions revealed competing ideas and led to a recognition of the need to know more about the context of a concept’s use in order to select the correct alternative (i.e., for rational numbers, we recognized that there are multiple legitimate meanings, and the context often is needed to determine the appropriate selection). And finally, research on misconceptions often led us to examine the historical development of an idea, only to discover that many of the competing ideas retained a worthy and still debatable co-existence (alternative conceptions), establishing that an enduring concept’s victory could be the result of culture, convention or logical primacy (e.g., Shapin and Schaffer’s, 1989, *Leviathan and the Air Pump*). Overall, misconceptions established clearly that learning was not a simple and direct accumulation of ideas and beliefs, with simple correction and replacement, but that its course would be circuitous, demanding revisiting and revising ideas as they gained intellectual breadth and power, and requiring careful attention to learners’ thoughts and perceptions. It further signaled that one’s view of epistemology and one’s philosophy of intellectual development in mathematics could be seen as relevant to an understanding of learning. Within the tradition of misconceptions, one critique which developed was that use of the term, misconceptions, focused too exclusively on the way in which student ideas deviated from traditional ones; and hence some researchers preferred the term, alternative conceptions, to signal the potential viability of their ideas (Driver & Easley, 1978). This emphasis on understanding the potential in student reasoning foreshadowed some of the developments of constructivism.

The third, and surely the most influential, tradition shaping the development of constructivism was the work of Piaget on theories of cognitive development. Whether Piaget’s work belonged to constructivism, agreed with constructivism, or defined constructivism is a matter of some debate (Ernest, 1991a; Von Glasersfeld, 1982). His prodigious writings and research production make it inevitable that he changed over the course of his lifetime that his co-workers and disciples often expressed varying perspectives, and thus individuals each interpret the work of Piaget. In our review of Piaget’s work, we recognize seven major contributions: (1) A child’s view is different qualitatively from an adult’s, (2) General stages of development viewed as likely intellectual resources for building ideas occur sequentially and provide important background information for studying children, (3) The development of an idea determines its meaning, rather than a simple statement of a formal definition and set of relationships, known as genetic epistemology, (4) Because of the first three premises, one can witness two major
kinds of encounters and responses to new ideas and information which are assimilation and accommodation, and (5) The process of moving from action, to operation, to mathematical object, required a level of consciousness that he labeled reflective abstraction. (6) The patterns of thought available for reuse and modification were cast as schemes, and (7) Describing foundational ideas often involved a search for conservation and invariance. These contributions due to their theoretical force and their connections to many replicable results through relatively simple experimentation stimulated the field to pay close attention to how fundamental concepts could be viewed as developmental and children could be rich resources of information. Added to the way it made one wonder about alternative ideas and ways children were not just incomplete mathematicians, the work of Piaget undergirded many of the constructivist activities.


If constructivism was forged out of these three elements, problem solving, misconceptions literature and Piagetian thought, one might ask, what catalyzed the movement as a whole, how was the whole greater than the parts, and what progress and/or limitations followed from the reformation of elements? In answering these questions, we will engage in our own form of “genetic epistemology” applying it to constructivism itself in relation to mathematics education over the last thirty years. We ask the question, what did the theory permit us to do, and where were its resistances, in the sense of the idea of viability or fit, rather than match, as proposed by Von Glasersfeld (1982)? Further, in the tradition of Lakatos, what we will be telling is a form of a “rational reconstruction” over the thirty-year history of PME, not a straight bibliographical retelling, but a re-positioning of the role of constructivism in the history of mathematics education as a response to a problematic. Further, we will ask the question of whether at the current time, it constitutes a progressive, degenerating or static research programme.

The problematic that constructivism sought to address was described at the sixth PME-NA meeting by Confrey (1984) as (1) Rigidity and limited student knowledge, (2) Excessively formal knowledge, isolated from experience and sense-making, (3) Dependence on external sources for evaluation rather than self-regulation; and (4) Emotionally intimidating and alienating. There was a clear focus on students and their perceptions of mathematics. In our approach then we will first outline the critical issues which emerged from the seminal meeting of PME in which constructivism was the theme. Then we propose our own framework of ten principles which we believe more completely captures the key elements of the theory. Our work on this can be viewed as a form of rational reconstruction in that we have selected these ten for their explanatory value in describing what is common and what is variable among the various interpretations of the theory. Because we cast constructivism as a grand theory, we see these ten principles as the hardcore and then suggest that one way to view the development of constructivism in PME is to report on the various bridging theories which linked constructivism into the practice of schools. These theories varied by region, by
content area and level, and by the aspect of educational practice selected. We present examples of these bridging theories in order to do some modest amount of justice to the breadth of work undertaken in the field. Finally, we discuss the overall directions in which constructivism has progressed and those in which more emphasis is needed.

In our reconstruction, we took a close look particularly at the watershed meeting in Montréal in 1987, as a means to consider what was attractive, what was controversial and disputed, and what was consensual. While the theme of the conference was constructivism, two of the plenary sessions were given by constructivist scholars Sinclair (1987) and Vergnaud (1987) whilst two were devoted to critiques (Kilpatrick, 1987; Wheeler, 1987). It was notable that two of the scholars whose works were continuously referenced and critiqued, Ernest von Glasersfeld and Leslie Steffe, were not provided an opportunity at the podium. We summarize their contributions as part of this section, recognizing them for their seminal work articulating clear, albeit controversial, versions of the theory. Their students, Cobb, P. Thompson, and A. Thompson among others, became some of the most influential scholars in developing the subsequent theoretical work.

Hermine Sinclair, a lifelong colleague of Piaget, gave the opening plenary, in which she expressed the view that the central tenet to Piaget’s work was “the essential way of knowing the real world is not directly through our senses, but first and foremost through our actions” (Sinclair, 1987, p. 28). Actions were defined as “behavior by which we bring about a change in the world around us or by which we change our own situation in relation to the world” (ibid, p. 28). She proceeded to describe how this constituted not just a learning stance, but an epistemological stance, in which one builds the cognitive structures that are needed to “make sense of experience”. In order to do so, she stressed the importance of the learner’s theories and successive models, that seek to know an object “which continues to possess unknown properties” (ibid, p. 29). She pointed out that constructivism represented a challenge to the Platonist view, noting that “to many adults, scientists as well as laymen, mathematical ‘truths’ appear to be a priori, Platonic ideas, that emerge at some point in development, whereas physical ‘truths’ are rooted in learning through experience, and thus fit into empiricist theories of knowledge” (ibid, p. 33). However, she also acknowledged that while rejecting Platonism, Piaget argued that logico-mathematical knowledge still differed from scientific knowledge. She discussed the importance of correspondences (and comparison) and transformation as the two primary instruments or processes linking the human subject and the objects of his knowledge. The starting point is through actions that become transformed into operations and processes. Then she emphasized that the construction of schemes from these interactions is a slow and gradual process, one which is “deeply rooted in all human endeavors to make sense of the world” (ibid, p. 35). She ended with a challenge to the audience to consider what other intellectual resources might explain the construction of mathematics beyond counting and possibly measuring.

In contrast, Jeremy Kilpatrick’s plenary was also identified reasons for the constructivist movement in mathematics education, and harshly attacked against
A 30-YEAR REFLECTION ON CONSTRUCTIVISM

aspects of it. He cited Von Glasersfeld’s description of radical constructivism as entailing two principles: (1) “Knowledge is actively constructed by the cognizing subject, not passively received from the environment”, and (2) “Coming to know is an adaptive process that organizes one’s experiential world; it does not discover an independent, pre-existing world outside the mind of the knower” (Kilpatrick, 1987, p. 7). In the next section, we discuss why this synopsis of constructivism left the field open to the debates that followed.

Constructivism was, for Kilpatrick, relegated to a metaphor, which responded to the question of whether mathematics is discovered or created (constructed). He suggested that the term ‘construct’ finds a particular resonance in mathematics where, in contrast to physical science concepts, one sees mental constructions as necessary for abstractions which eventually lack a material referent. If the metaphor of construction is solely to apply to a means to maximize students’ involvement, engagement in mathematical activities, then he found it palatable. If, instead, it entailed a rejection of an external, knowable reality, objective truth or Platonist views of mathematics, he opposed it. He claimed that one could take a step into a development perspective and the evolution of child thought, without stepping into epistemology, which entailed redefining or reconstituting mathematics itself. Two imperatives of some versions of constructivism were objected to are: (1) The description of humans as self-organizing, based on their responses to perturbations (which he described as based on “negative feedback or blind” and “closed” (Kilpatrick, 1987, p. 9), (2) The limitation of knowledge to issues of epistemology rather than ontology, the study of what is in the sense of “being” or “reality”. Finally, he required that constructivism pertain only to learning, hence rejecting Von Glasersfeld’s described consequences of constructivism for educational practice. Furthermore, he argued what constructivism needed to do was to accept the views of mathematics provided by mathematicians, such as Ruben and Hersch, accepting math as a Platonist enterprise, only explaining the behavior of mathematicians as a socio-cultural artifact. Kilpatrick’s concerns were subsequently shared or discussed by others (Ernest, 1991a; Goldin, 1989), hence our careful treatment of it is included.

The pressure on the two discussants, Vergnaud and Wheeler, was palatable for those at the meeting. Gerard Vergnaud began with a clear challenge, stating “As a matter of fact, our job, as researchers, is to understand better the processes by which students learn, construct or discover mathematics and to help teachers, curriculum and test devisers, and other actors in mathematics education, to make better decisions. This is our practical burden” (Vergnaud, 1987, p. 43). Vergnaud recognized that epistemology was a key part of Piaget’s work, and pointed out that for Piaget, “constructivism contradicts both empiricism and a priori rationalism” (ibid, p. 44). He reminded the audience that Piaget’s work on space and time rejected Kant’s claim that these two constructs transcended human knowledge. Piaget rejected Hume as well, based on the view that it could not be assumed that knowledge comes directly from sensory perception. Vergnaud attributed Piaget’s brilliance to finding a way to weave together empiricism and rationalism, which
Vergnaud argued “must be probably traced in his background as a biologist and an evolutionist”. (ibid, p. 44)

This emphasis on adaptation was key for Vergnaud (1987) as children ...

... have schemes and categories to interpret experience, and that these schemes and categories are not a priori schemes and categories but derive from inborn schemes and experience. Action is essential as children accommodate their schemes through action upon the physical (and social) world, in order to assimilate new situations, nearly in the same way as scientists develop new procedures and concepts from former knowledge to understand and master new phenomena. (p. 44)

Vergnaud then carefully distinguished his position from radical constructivism, premised on his view that radical constructivism entails a “denial of an independent pre-existing world” and “fails to provide a theory of objective knowledge” (ibid, p. 46). He simply stated that “Piaget was not interested in this metaphysical question” (ibid, p. 45). While we would argue that both of these attributions to radical constructivism are incorrect in our understanding of it, his clear negotiation of the disputes on both sides was a feat of diplomacy and scholarship.

The final section of Vergnaud’s talk concentrated on mathematics, which he emphasized, had not been sufficiently addressed in the previous talks. He questioned the cogency of Piaget’s distinction between empirical abstraction and reflective abstraction, offering instead his concept of “theorems in action” (Vergnaud, 1987, p. 47) which he claimed must precede the development of formal theorems. By arguing that in the beginning, for children, physics and mathematics are indistinguishable, as quantity and measure both require ideas of space and time, he suggested that this level of challenge of the distinction was needed to overcome the view that mathematics was about “additive rules, symbolic calculus and static structures” (ibid, p. 47). At the same time, he acknowledged that the development of the irrational number and the need to move towards an understanding of “pure numbers” symbolized places where mathematics would depart from its physical roots.

Wheeler’s final plenary veered back towards the critical. Renouncing constructivism as a theory, he wrote, “It is not a theory because it is not formulated in terms that could lead to refutation. At the heart of discussions about constructivism is the difficulty that its espousal and its rejection are more products of taste than of evidence” (Wheeler, 1987, p. 56). What Wheeler did was to suggest that constructivists were mixing together uncritically a number of elements: (1) Constructivism as practiced in mathematics (of which he wrote, “The platonist-constructivist dichotomy puts us in the position of either denying that we have any choice in the directions in which mathematics develops or deny that the inner coherence of mathematics ever takes us in directions different from those we intended to follow” (ibid, p. 57)); (2) Constructivism and psychology (where he cited Blakemore, 1973, who stated, “our seemingly unified view of the world around us is really only a plausible hypothesis on the basis of fragmentary
evidence” (ibid, p. 57) and stressed the impact of “a priori powers of the mind or the brain that enable us to select invariants form the flux of sensory data” (ibid, p. 57)); (3) Constructivism and philosophy (he noted Piaget’s contribution to the philosophy of mathematics by stating “Piaget’s constructivism, meaning is cumulative and the evolution of mathematical structures is towards increasing comprehensiveness and rigor. Logico-mathematics structures build on those that came before, integrating them while overcoming their inadequacies. Mathematics therefore moves towards increasing objectivity—which Piaget understands as a process and not as a state” (ibid, p. 59)); and, finally, (4) Constructivism and education where he acknowledged the importance of linking knowledge to the intentions of the learner while respecting the context of school’s need for organized and generalized knowledge. He too acknowledged a place for students “to see themselves as originators and modifiers of knowledge”. (ibid, p. 59)

As indicated previously, the contributions of Ernst von Glasersfeld and Leslie Steffe to constructivism were of considerable theoretical importance. Von Glasersfeld generally took a philosophical approach he termed “radical constructivism” which he linked to Aristotle, Vico, Dewey, and James which classifies “knowledge is the result of a learner’s activity rather than of passive reception of information or instruction” (Von Glasersfeld, 1991, p. xiv) and that therefore he argued that knowledge should be conceived of as an “adaptive function [which] ... means that the results of our cognitive efforts have the purpose of helping us to cope in the world of our experience, rather than the traditional goal of furnishing “objective” representation of the world as it might “exist” apart from us and our experience” (ibid, p. xv). This position led him to articulate a fundamental distinction between “fit” and “match” where he drew an analogy to a lock. He pointed out that many keys could potentially fit a lock and hence knowledge was more akin to devising a possible key than creating a mirror image of the lock itself or “matching it” (Von Glasersfeld, 1982). Steffe’s contributions were both theoretical and methodological. Besides the careful analysis and set of distinctions Steffe, Von Glasersfeld, Richards and Cobb (1983) offered on how children developed counting and operations, Steffe and his colleagues also wrote numerous articles concerning the methodologies of teaching experiments and clinical interviews which describe how to build a model of children’s mathematics (Cobb & Steffe, 1983; Steffe & Thompson, 2000).

Steffe’s subsequent distinction between first- and second-order models provides a way to consider the role of social interactions explicitly both by the interviewer and when observing students’ and teachers’ interactions (Steffe, 1995). His critical contribution was his articulation and illustrations that knowing how other’s conceive of mathematics is a challenging enterprise and must be based on extensive observations with carefully sequenced tasks: “As a teacher, one must intensively interact with students to learn what their numerical concepts and operations might be like, and how they might modify them as they interact in situations of learning” (ibid, p. 495). He suggested that the first-order models were a means to describe subjective mathematical experience. To build on an issue raised by Thompson who argued, “One notion that I resist is the notion of social
cognition ... it is only in the mind of an observer that socially constructed knowledge is ‘out there’ ... and ... as a consensual domain” (ibid, p. 496), Steffe proposed the idea of a second-order models. These were used to explain social interactions which are “necessarily constructed through social interaction because they are the models that the observer constructs of the observed.” (ibid, p. 496)

Looking back, we are struck by the sense of incompleteness of the session, and suggest that the field was not yet ready to negotiate a way to settle the disputes. In our rational reconstruction, we locate the heart of the problem in the statement of constructivism’s two principles, which led to a view that if one accepted only the first, one was cast as a trivial constructivist and if one accepted the second together with the first statement, one was a radical constructivist (Von Glasersfeld, 1995). Many mathematics educators faced with a choice steered clear of the controversy and hence never confronted the subtler implications of the constructivist theory. As a result, we would argue that this debate distracted from the essential point in constructivism which is to recognize the profound impact of theories of evolution on intellectual work. It turns out that all three major theorists, Piaget, Dewey, and Vygotsky were addressing these implications to various degrees. As Dewey recognized in *The Influence of Darwin on Philosophy* (McDermott, 1981),

That the combination of the very words *origin* and *species* embodied an intellectual revolt and introduced a new intellectual temper is easily overlooked by the expert. The conceptions that had reigned in the philosophy of nature and knowledge for two thousand years, the conceptions that had become the familiar furniture of the mind, rested on the assumption of the superiority of the fixed and final; they rested upon treating change and origin as signs of defect and unreality. In laying hands upon the sacred ark of absolute permanency, in treating the forms that had been regards as types of fixity and perfection as originating and passing away, the *Origin of the Species*, introduced a mode of thinking that in the end was bound to transform the logic of knowledge, and hence the treatment of morals, politics and religion. (p. 32)

**PART 3: TEN PRINCIPLES OF CONSTRUCTIVISM**

As we reflected back over the 1987 Montréal meeting and subsequent work, we conjectured that the concise and elegant two-principle statement may have contributed to perpetuating and polarizing the debate. It reduced the complexity of the epistemological import of constructivism to a single statement too easily misunderstood. Partly as a result of this and as a result of the lack of precise specification of the mathematical implications of the theories, constructivism was frequently associated with excessive student-centeredness, lacking deep enough attention to the role and value of established mathematical accomplishments and proficiencies.

This analysis led us to restate the theory in terms of ten principles selected for their explanatory potential to help to highlight the characteristics of the subsequent
work on constructivism by the international PME community. As a “rational reconstruction” of the thirty years, one seeks to explain both the commonalities and variations in the theory as it evolved in a variety of locations, with concentrations of different age groups, topics, and forms of instructional practice. This articulation is also intended as our own contribution to subsequent work in the area. Thus, we are suggesting that a more complete and satisfactory articulation of the principles of constructivism might be stated as including the following:

1. An *explanatory* model for development is necessary to guide educational practice. A descriptive model of stages is insufficient as it will only tell one what behaviors to look for, and not how to achieve them. An explanatory model is needed identifying processes for change as well as likely paths of change over the course of learning.

2. Since evolution and adaptation provide a convincing model for conceptual-historical evolution of ideas (phylogeny), a strong candidate for articulating an explanatory model and underlying mechanism for development (ontogeny) is likely to reside in identifying parallel constructs. Likewise, it would need to explain variation, similarity, change over time, and selection. *Genetic epistemology* is such a theory as it seeks to explain the ontogeny of intellectual development in terms of an individual’s interactions, both social and environmental. It changes our focus from classical epistemology where we concentrate solely on the products of knowledge and their justification abbreviated in the phrase “justified true belief” (or what we know and why we believe it). In addition, it focuses our attention on how we come to know it (processes) and how we communicate that knowledge with others (social interaction). This principle of constructivism does not require one to reject ontology, or an external reality or existence, but only to recognize and focus on our ongoing active participation, by means of tradition, practice, and physiology, in the process of knowing. Accepting that as an organism, our ways of interacting shape what we claim as knowledge, does not obligate one to reject the view that things independent of us shape those possibilities and action. The debate concerning the relationship between reality and knowledge still flourishes in some circles, especially concerning what legitimate appeals are for warrant and the meaning of truth. However, our treatment of constructivism emphasizes the ideas of viability and fit (Von Glasersfeld, 1982) rather than of permanent truth and assured objective properties. Fallibilism in epistemology (Ernest, 1991b) was mistaken for solipsism in ontology by many critics of constructivism. Rather, constructivism seeks to steer a course between positivism and solipsism. As stated by Larochelle and Bednarz (1998), “Escaping the dictatorship of the object – the position of naive empirico-realism – only to come under the rule of the subject is not a particularly innovative solution” (p. 5). Genetic epistemology focuses our attention on creating an explanatory theory which elaborates “a theory of the organism who creates for him- or her-self a theory of the world” (Von Glasersfeld, 1987, cited in Larochelle & Bednarz, 1998, p. 5). It concerns how and by what means an individual determines what theories of the world “fit” his/her experience writ large (including social and environmental factors) rather than to decide to what extent these theories “match” an external reality, hence the stress is on
epistemology. Emphasizing adaptive fit requires a rejection of a correspondence theory of truth, which then needs to be replaced by alternative ways of linking human activity and the world to produce and explain forms of warranted knowledge. Two such approaches are described in principles 3 and 4.

3. Truth can be obtained in relation to a coherence theory of knowledge within the mathematical practice of building axiomatic systems, if it serves the role of establishing consistency within a limited system. That is, one accepts the “truth” of statements that are derived deductively from axioms taken as starting points. While some may prefer to call it truth, others may prefer the term “certainty” (Von Glasersfeld, 1990) in recognition of the fact that even rules produce ambiguity and the need for further refinement of the terms, definitions and scope of applicability. Coherence alone, however, is not sufficient as a lone explanation of truth because of the incompleteness of axiomatic systems to describe all of mathematics and hence even with coherence, one still needs to also consider other sources of warrant. Furthermore, one will need to consider the balance of attention to be paid to these multiple sources of warrant and how an understanding of coherence is developed.

4. In mathematics, warrant also derives from the careful development of conjecture, argument and justification concerning the study of number, space, pattern, change, chance and data. We refer to these processes as chains of reasoning which are the hallmark of mathematical thought, and they include intuition, visualization, generalization, problem solving, symbolizing, representing, demonstrating and proving etc. In these areas, constructivism attends to how actions, observations, patterns, and informal experiences can be transformed into stronger and more predictive explanatory ideas through encounters with challenging tasks. These ideas or concepts can then become tools for building new concepts within each of these subfields. While deductive reasoning is certainly one important aspect of this (discussed in principle 3 with coherence), constructivism recognizes the value of other forms of securing mathematical certainty such as the coordination of representations, the identification of patterns, the recognition of similar ideas in apparently dissimilar settings (connections), the development and refinement of conjectures, and the applications of the ideas to other fields. This myriad of mathematical concepts and processes retain their connections to everyday experience, hence replacing the need for correspondence with the satisfaction of purposeful activity to resolve outstanding problematics.

5. We select the individual as the primary unit of analysis for assessing and evaluating cognitive achievements in acknowledgement of the need to ensure that the complete patterns of reasoning associated with key ideas are understood at the individual level with associated coherence, adaptive fit and continuity. This is akin to Steffe’s first-order models, and does not imply neglect of the ways in which those experiences are nested and shaped within patterns of participation in larger collective membership units (dyads, classes etc.). It is further a practical decision based on typical schooling, which treats students as individuals as they move across grades, across locations, at the level of assignment, in relation to future studies and work, and in relation to the basic accountability systems. We recognize
the importance and viability of also including other units of analysis, such as dyads, groups, classes, schools etc., as a second-order model in relation to the assessment of an individual’s developmental path. The distinction between first- and second-order models will prove useful to the observer/researcher, but should not lead one to assume that the individual student experiences them as separate. We liken this decision to place the individual as the first-order model to Vygotsky’s choice of the word as the fundamental unit of analysis which did not preclude his theorizing about sentences or complex social interactions, but it guided his empirical designs and permitted him to identify the building block of his theory. Likewise, constructivist scholars investigate collective social interactions, purposes and forms of engagement, and coordinate these with students’ interactions with various physical devices and tools, but our claim is that collective social interactions should be linked with its effects on individual student’s intellectual growth. Further this should not be construed to mean that personal identities are considered only as individually constituted, nor does it imply that membership in multiple groups is neglected or ignored.

6. To explain sources of variation for individuals and avoid a standardized or uniform theory of knowledge, one needs to consider three broad and interacting factors: The individual’s current state of development, social and cultural influence as members of a tribe (group), and environmental/physical factors in relation to the task at hand. While in evolution, mutation is the primary source of variation, we rather ascribe unique arrangements of the three interacting factors as the means of producing the essential diversity that spawns invention and serves as a source of variation. One of the most compelling contributions of constructivism is the documentation of rich and interesting ways that children express about ideas. We see it in the form of inventive representations, language, forms of reasoning, alternative pathways, and explanations. Many of these expressions are regularly overlooked in traditional classrooms. This can result in missed opportunities for interesting connections among ideas, can undermine children confidence in their own emerging reasoning, and result in proposals which are labeled as erroneous that may support alternative paths.

7. To explain selection, one must consider how the same three forces act to define criteria for viability for cognitive ideas, (as mortality vs. survival would not serve this purpose). First, we point out that selection depends on processes of change and adaptation. We propose that pragmatism, in relation to functional fitness, provides the means for this; that a difference is viable when it makes a difference (James, 1907). This conception then invites one to propose sets of processes that instigate, regulate, and evaluate change in terms of functional fitness. For Piaget, these were assimilation and accommodation. For Dewey, it was the process of inquiry, wherein the indeterminate situation is transformed to a determinate situation. For Peirce, the stress was placed on the importance of doubt in securing deep understanding (see Peirce, 1877, 1878). In constructivism, compatible with both of these philosophers, cognitive change, or intellectual growth, begins with a perturbation, or a problematic, which is a perceived roadblock to where one wants to be (Confrey, 1991). It is followed by an action, to
attempt to eliminate that perturbation or to satisfy the felt-disequilibration. As emphasized in Sinclair (1987), the action of the individual is key in that the degree of active participation often determines the success of the action in resolving the problematic. Also as she emphasized that action often involves comparison or transformation of the original situation. In most school-related settings, as well as many others, a representation is produced to record, signify or communicate the results of that action. This leads to and supports an act of reflection, to assess whether the original perturbation or felt-need was satisfied, or whether more action is required. The cycle repeats itself, continuing to transform the problematic hopefully towards resolution. This cycle of constructive activity represents the activity of selection for viability of ideas. In all steps, to varying degrees, the influence of social and environmental factors are at play – sometimes with more or less emphasis on one or the other. Summarizing this process, Larochelle and Bednarz (1998) wrote,

Drawing on a range of fields including second order cybernetics and contemporary linguistics and epistemology, constructivism centers of the development of a “rational” model of cognitive activity of either an individual or collective variety, including the narratives which are devised to give shape and meaning to our actions ... Or, to take Korzbsky’s metaphor, a map can never be said to “be” the territory – all the more so in that the territory is a question of representation as well. What the map refers to inevitably an affair of not only the particularities decided on by its maker but also the distinctions he or she chooses to make in accordance with his or her project and the success with which his or her cognitive and deliberative experiences have met. (p. 6)

8. In learning, there is an unavoidable element of recursiveness in the process. One recognizes multiple forms of awareness of oneself as a learner – as one: (a) Determines if the goal, purpose or problematic has been satisfied; (b) Creates records and representations to communicate with others and/or to assist in reflection and evaluation, and (c) Remembers successful and viable methods for future use (schemes). In addition, in the description of learning, the levels of recursiveness accumulate further. As stated by Von Foerster (1984), “it takes a brain to write a theory of the brain; now, for this theory to be complete, it should also be able to explain the fact of its own elaboration, and what is more, the writer of this theory ought to be able to account for his or her writing” (p. 11). Properties of the observer must be part of the description of what is observed (Larochelle & Bednarz, 1998). That is, our explanations must serve to both describe what we observe and to explain our own experience, at the level of mechanism. It is this recursiveness that produces in humans the particular ability to abstract, a key element of mathematics.

9. Because in constructivism, the focus is on genetic epistemology, objectivity must be redefined as the result of a consensus among a group of qualified individuals to authorize a particular description or explanation as viable and as shared among them. According to the standards of any particular set of knowledge
games (discipline), the standards for authorizing knowledge differ, and as a theory about functional fitness, objectivity represents a perceived stability in ideas, not a permanent state of being. This is more akin intersubjectivity as discussed in Thompson (2002). It can be a case of a symmetric assumed tacit understanding by all parties as Cobb’s “taken-as-shared” (Cobb, Yackel & Wood, 1990), or a case of a stated and negotiated understanding or asymmetric but uncontested recognized difference by one or more parties, as Confrey’s “agreeing to agree” (Confrey, 1995). How these bear upon and are used in the development of an individuals’ independent reasoning in mathematics or science is a source of valuable investigation and has led to the development of socio-constructivism as a distinct subset of constructivism. Within such an approach, one can examine the development of “knowledge communities” as a larger unit of analysis, provided it is connected to its effects on independent reasoning patterns for individual students, as also a target unit of analysis.

10. An understanding of the first ideas will lead people to more viable and effective models of knowledge and will engender more productive knowledge acts as one recognizes the observer-observed interactions not as limitations but as accomplishments and agreements, and not simply received knowledge, but as active choices and selections by reflective knowers or consciousness. This treatment of consciousness should be a primary outcome of learning in science or math. Désautels (1998) recognized the need for a broader level of awareness than what is obtained by reflective abstraction in terms of understanding by jumping to a recognition of how these chains of reasoning are embedded in a larger framework of knowledge construction and debate:

One is justified in thinking that ignorance of the relative, discontinuous, and historically located character of the development of scientific knowledge (Serres, 1989) will leave this student quite unprepared to gauge the limits of this type of knowledge and to appreciate the real worth of other knowledge forms and knowledge games. (p. 124)

Whence the necessity, if one wishes to participate in the conversation of scientists, of understanding how the latter impart meaning to the notions and concepts they use; whence also the importance of epistemological reflexivity. Only when knowing subjects become aware of the postulates which underlie their usual ways of knowing, and when they place their own knowledge, they will become able to open themselves to other potentialities. Although the intellectual process of reflexivity is often associated with metacognition, it is distinct from the latter in that it does not involve the intellectual operations or strategies in developing this or that bit of knowledge. Instead, reflexivity draws attention to “that which goes without saying” – that is, the unspoken assumptions or the un-reflected aspects of thought which lead one to be referred to metaphorically as the blind spot of a conceptual structure which is a condition necessary for beginning that process whereby thought is complexified and autonomized (Varela, 1989). (p. 128)
JERE CONFREY AND SIBEL KAZAK

This restatement and elaboration of the premises of constructivism into ten principles simultaneously accomplishes two goals. First, it rejects the dichotomy between radical and trivial constructivism, arguing instead for a more nuanced set of distinctions. Secondly, it draws upon the contributions of each of the scholars cited previously. It is consistent with the statements by Sinclair (1987) about the centrality of action and the multiple roles of reflection. It responds to the criticisms of Kilpatrick (1987) by clarifying the atheoretical position of constructivism on ontology and by redefining objectivity within a social constructivist perspective. It links constructivism to its philosophical basis as demanded by Wheeler (1987), and demands that the psychological view of constructivism recognize the epistemological central hardcore of the theory. And finally it avoids the criticisms of creating an overly individualistic or solipsistic theory.

While this revised statement is consistent with the arguments by Vergnaud (1987) and his criticisms of his interpretation of constructivism, it does not accomplish his primary challenge, which is to use constructivism to explain mathematical knowledge and instructional practice. Nonetheless in our rational reconstruction of the constructivist research programme, we believe it prepares the way more adequately for doing so, as it restricts the theory’s scope of application to reasoned knowledge and locates the coherence of evolution in the individual’s students’ minds, while recognizing the significant forces exerted by other types of knowledge and other units of analysis. Thus, in the remainder of the paper, we address how constructivism has affected our understanding of how children learn the concepts of numeration, quantification, space, logic, chance, change, and data.

PART 4: MAJOR ENDURING LEGACIES OF THE CONSTRUCTIVIST RESEARCH PROGRAMME (1996-PRESENT)

Because we do not see the contributions of constructivism as only theoretical, but also specific and practical, we have identified nine major enduring legacies of the constructivist research programme. We select a few examples from each to illustrate how these provide the “protective belt” around the constructivist core principles and support the empirical evidence to link it with practice. One possible exercise, beyond the scope of this paper would be to consider how each of these bridging theories draws upon or modifies the ten fundamental principles of constructivism as previously outlined.

Bridging theories

As a grand theory, or perhaps a paradigmatic theory, constructivism is too general to reach to the classroom directly. This gap is accounted for in different ways by different people. Some say it is because it is a theory of learning rather than of teaching (Simon, 1995). Others say that it is not specific enough to mathematics, or perhaps particular subfields (geometry, multiplicative structures, etc.). For others, the missing elements are the artifacts of practice –curricula, technologies or
assessments. In this first section, we describe how members of PME brought to our community means of linking constructivist theories into practice.

Examples of bridging theories in PME are numerous. We would point to examples in the work at the Freudenthal Institute on Realistic Mathematic Education (RME) (De Lange, 1987, 2001; Gravemeijer, 1994, 2002), the work on additive and multiplicative conceptual fields (Harel & Confrey, 1994; Steffe, 1994; Vergnaud, 1996), theories of advanced mathematical thought (Sfard, 1991; Sierpinska, 1990; Tall, 1991), didactical engineering (Artigue, 1987; Brousseau, 1997), modeling and applications (Blum, 1993; Burkhardt, 1981; Niss, 1992), the theoretical work of Pirie and Kieran (1994) (descriptions of stages of concept development), and cognitively guided instruction (CGI) (Carpenter et al., 1999) to name only a few.

RME was both an approach and a set of curricular materials. Freudenthal, beginning in the 1960s, had invented the idea of mathematization with two components: Horizontal and vertical. Horizontal mathematization was “where students come up with mathematics tools that can help to organize and solve a problem set in a real-life situation” and vertical mathematization “is the process of reorganization within the mathematical system itself” (Van der Heuvel-Panhuizen, 1999, p. 4 cited in Perry & Dockett, 2002, p. 89). These initial distinctions led to the development of design heuristics that included “guided reinvention” and “didactical phenomenology”, both of which provided a genetic aspect to the instructional approaches and worked to capture the need for students to strengthen their understanding of abstract ideas while linking them to practices involving the application of quantifiable knowledge. De Lange (1987) followed with the development of a new curriculum focusing on applications and assessment for upper secondary mathematics. In recent times, Gravemeijer (1999) has further extended the work to include the ideas of “emergent modeling”. In all of these efforts, we see clear links to the constructivist principles including an explanatory theory of what mathematics represents, how students’ move towards increasing proficient uses of symbolization, the importance of reflection in that process, within social and interactional settings, and the importance of distinguishing teacher and student perspectives and make the observer’s position one of problematizing and gathering evidence on students’ perspective.

The area of advanced mathematical thought demonstrates another example of a bridging theory between constructivism and classroom practice. The theories in this area arose from the recognition that many students demonstrate a gap between their informal and formal knowledge, having learned to correctly reproduce the formal definitions, but retaining contradictory commitments in their informal knowledge. Beginning with the work of Tall and Vinner (1981) distinguishing concept image and concept definition, researchers in this tradition tried to reestablish the roots of complex ideas in constructivist beginnings. For instance, Vinner’s work on concept images in functions demonstrated that while students could recite the formal definition of functions with some accuracy, they still reasoned with specific cases in ways that were not consistent with that definition (Vinner & Hershkowitz, 1980). For instance, when presented with a
JERE CONFREY AND SIBEL KAZAK

graph in pieces, students reasoned that it could not be a function, though it met the formal criteria. Ascertaining what students think and reasoning what the implications of those responses are critical elements of constructivist perspective.

This research evolved into producing Advanced Mathematical Thinking (Tall, 1991) and a PME working group that continues to the present. It included Douady’s dialectique outil-object (Douady, 1986), Sfard’s dual nature of conceptions (Sfard, 1991), Gray and Tall’s procept (Gray & Tall, 1994), and Dubinsky’s APOS (action, process, object and schema) (Dubinsky & McDonald, 2001). All sought to explain how to move from contextually situated, action-oriented ideas to increasing levels of abstraction. They sought to explain the development of reification, where in mathematics, an idea used at one level, becomes an object on which to act at the next. While some criticized the linearity of the approach (Tall, 1991) and the narrow focus on abstraction as the absence of context (Confrey & Costa, 1996), the research had strong ties to constructivism, particularly in its use of reflective abstraction as a means to bootstrap into advanced thinking, and it helped instructors to learn to pay closer attention to students’ thinking in the building of mathematical ideas. To a degree, it broadened the views of mathematics, emphasizing the need at all levels to consider the role of student conjecture, reflection, and development. These major contributions of bridging theories by scholars typically entailed most of the ten principles as they worked out extensive ways to create curricula or to influence instructional programmes at various levels.

Grounding in action, activity and tools

A second common element of most of the constructivist initiatives in mathematics came from the claim that mathematical ideas are fundamentally rooted in action and situated in activity. During the thirty years of PME, we witnessed researchers from around the world prospecting the sources of mathematics ideas, in a variety of ways. One generative route came from the ways in which performance on everyday tasks contrasted with performance on school-based or more formal tasks. While this research tradition evolved into situated learning (Brown, Collins & Duguid, 1989; Greeno, 1989) and ethno-mathematics (Carraher & Schliemann, 1988; D’Ambrosio, 1985), and typically attached themselves to socio-cultural perspectives as the “grand theory”, many of the ideas served as the basis for conceptual development within the constructivist tradition as well. For example, research on candy sellers revealed a different form of primary units that base 10, which at 35 cents a candy bar, tended to group to $1.05 as a unit. Besides demonstrating again the differences in character of formal and informal knowledge, it taught researchers to look for competence in formally less educated clients, rather than to assume only formal knowledge was productive and accurate. We still see this heritage in developmentally early curricular tasks focusing on situational units (silhouettes of footprints, handprints) as measurement units.

This research led researchers to explore and validate student strategies and approaches (Ginsburg, 1989; Kamii, 1985). Ginsburg began his work with the
observation from extensive clinical interviews that children’s thinking is seldom capricious. Kamii used the Piagetian approaches to develop a variety of ways to use manipulatives and related materials to build young students’ understanding of arithmetic; and she paid careful attention to how to transition from the material actions and operation to notational and symbolic use.

The focus on action as the source of mathematical ideas spread throughout mathematics education beyond the use of manipulatives and everyday objects to consider the potential generativeness of a variety of tools. One potent source of this was by researchers interested in the development of algebra and the concept of function, who located this work in the curve drawing devices of the seventeenth century (Bartolini Bussi, 1993; Dennis, 1995; Taimina, 2005). Their work illustrated the potential of historical investigations that showed that algebraic descriptions of curves did not come in an f(x) format, where independent variable produced dependent variables. Even the Cartesian plane, as constructed by Descartes, was not dependent on perpendicular axes, but located them for convenience in describing the curve (Smith, Dennis & Confrey, 1992). Demonstrating the links between algebra and geometry through similarity and proportional reasoning, these scholars (Bartolini Bussi, 1993; Dennis, 1995) showed that the development of algebraic expressions for functions had their roots in a variety of tools for constructing different curves, well-beyond the traditional constructions with straight edge and compass to hinged devices. Later as dynamic geometry became available, it was used as a powerful representational media for exploring these ideas further. In it, students could experience how it was invariant properties of a class of curves, and not scale, that was the defining feature of families of functions.

Others extended this type of work to create new devices for exploring the sources of student reasoning (like Meira’s, 1995, gears, and Nemirovsky’s, 2002, trains), and motion detectors, and other devices provided potent sources of mathematical explorations.

More recently, a research forum, titled “Perceptuo-Motor Activity and Imagination in Mathematics Learning,” was organized at PME-27 by Nemirovsky and Borba in 2003 just to discuss how the devices such as water wheel, sensors together with graphing calculators and software such as LBM connected to mini-cars brought different experience students have outside the classroom into the teaching and learning of themes, such as middle school algebra, introduction to functions, calculus, and dynamical systems. This forum became recently a special issue of Educational Studies in Mathematics (Nemirovsky & Borba, 2004) which was published in a format of a video-paper, a multimedia artifact brings voice of students in a new way to mathematics education research. Not only the voice of students can be literally heard, but body language related to their interaction with standard mathematics representations (e.g. graphs) and to the artifact can be differently experienced by the reader-viewer.

We can definitely see the roots of constructivism in the examples presented in this section. They emphasize the notion of voice of the students and how it affects the perspectives of the teachers/researchers and how their thoughts are modeled by
JERE CONFREY AND SIBEL KAZAK

the others (Confrey, 1998). Differences in understanding of a given concept in differing contexts and situations has been a hallmark in this kind of research, as well as careful documentation of how different students develop and contribute diverse ideas as they interact.

It was a small but profound step from these physical environments to the application of new technologies as surrogates for concrete grounded activities. Best known was Logo (Papert, 1993), with its successor in Star Logo (Resnick, 1994) and Lego/Logo (Resnick & Ocko, 1991), which led to a closely related branch of theory called constructionism based on the concept of microworlds. In 1988, we saw the maturation of such research in the work of Hoyles and Noss (1988) on Logo programming environment. Nesher (1988b) tackled directly the issues of truth in mathematics when she developed the concept of learning systems suggesting that one can find “microworlds” that are nearly isomorphic to the axiomatic systems but which are held together by a more experiential glue that formal deductive reasoning and cites Logo microworlds as such an example. She argues that these provide a means of giving children the experience of the coherence view of truth.

Other environments that profoundly affected PME were Cabri-géomètre (Baulac, Bellemain & Laborde, 1988), the Function Supposer (Schwartz & Yerushalmy, 1988) and Function Probe (Confrey & Maloney, 1991) and spreadsheets (Sutherland & Rojano, 1993), SimCalc Math Worlds (Kaput, 2001), and quantification of motion in Thompson (2002). In all of these, we watched as physical actions on objects were transformed into machine-driven actions in which students became active investigators of the properties illustrated in the technologies and software. These software tools permitted students to link their opportunities for conjecture and exploration to the new century’s tools of mathematics.

We would suggest that these efforts constitute means to strengthen the meaningfulness of mathematics. In relation to our framework, we emphasize that the approaches provide examples of genetic epistemology; that they provide a means to see how correspondences between symbolic activity and everyday activity could be fostered within mathematical activity.

Alternative perspectives, student reasoning patterns and developmental sequences

One characteristic of constructivism is to offer adaptation as mechanism to explain the transformation of human thinking over time. Research in this area has included work on counting, ratio, statistics, probability, limits, functions, and geometric proofs.

This has led mathematics educators to identify critical moments in learning where an earlier way of thinking fails to account sufficiently for new ideas and where an invention is needed to account for those examples, extensions or phenomena (Nakahara, 1997). A number of these ideas are found in mathematics: Perhaps the best known is multiplication makes bigger, division makes smaller. Researchers recognized that extending multiplication and division to these values must be accompanied by encountering directly this conflict in expectations. Such
A 30-YEAR REFLECTION ON CONSTRUCTIVISM

an encountering is not simply a matter of seeing the result, but often of reexamining one’s underlying models. For instance, if multiplication is based in arrays, then multiplication by a fractional part will require a transition to area models. Further, if the problem of $a \times b$ for $a>1$ and $0<b<1$ is managed by using commutativity and repeated addition of the fractional unit, the next case where both $0<a<1$ and $0<b<1$ must still be managed instructionally. As this research evolved, we learned that even when the issue is “resolved” for multiplying $a/b \times c/d$, it resurfaces when students are faced with multiplying $3.45 \times 0.56$. Here, Greer (1987) examined what students predicted when asked to calculate prices of gasoline where flaps over the values obscured the numbers and how they changed operations when revealing the numbers faced this with values like those given previously. He argued that students should not be inclined to change their predictions if they possessed what he labeled “conservation of number” (Greer, 1987). This research tradition suggested that certain beliefs of children develop in limited settings, and that extending them in ways that conflict with those original predictions one not only to provide them with the new procedures, but to get them to think through why there is a need to revise their ideas.

This work has evolved into current research on students’ computational strategies and understanding where one considers the interplay among their understanding of number types and magnitude, operations, situations, units, and complexity of operations (number of steps; order of operations) (Verschaffel, De Corte & Vierstraete, 1999).

Similar work has been conducted in two areas: (1) Counting, addition, and subtraction, and (2) Fractions, ratio, and multiplicative structures. Research on counting and the development of adding units (Carpenter, Moser & Romberg, 1982; Steffe et al., 1983) commenced a string of work which culminated revisions to curricula around the world. Here, Vergnaud first presented his work on “theorems in action” and conceptual fields (Vergnaud, 1982). In the beginning, Noelting (1980) developed insight into a set of stages he observed in students learning ratio and proportion reasoning. Others (Hart, 1984; Streefland, 1991) extended this work to document the types of common errors made by students in rational number reasoning. One set of researchers then recognized that the student difficulty with these ideas was at least as likely to be a product of competing conceptions of rational number and identified six major parts. The Rational Number Project group (Behr, Lesh, Post & Silver, 1983), Nesher (1988a), Kieren, (1992) and Vergnaud (1996) discovered that it was not just an issue of the numbers but of the interplay between the numbers, their representation, the situation and operations, and properties that created a network of relations that one must move among to operate correctly, referred as a multiplicative conceptual field. It was in this context that Vergnaud developed further his observation that students were in fact drawing on implicit “theorems in action” (p. 225). By setting the work on multiplication and division in the context of exponential functions, Confrey and Smith (1989) argued that there were multiple legitimate concepts of rate. It can be seen that this research programme had a clear constructivist tenor— it recognized the pragmatist roots of different conceptions and the lack of distinction inherent in
the symbolic form of a/b. It began with an expectation of a single conceptual development trajectory and quickly found that in fact, a number of different concepts are in conflict with each other and that while a single consistent formal structure can be imposed on these, the need for distinctions continues as students work in context.

The work on developmental sequences, undertaken in many areas of mathematics, geometry, statistics as well as those outlined above, have contributed key insights into student reasoning. Over the years of constructivist research, we have come to realize that the success of learning depends on a careful but flexible sequence of activities which adapt to student ideas while encountering critical barriers. Much productive work needs to be undertaken to test these sequences at large scales with more carefully designed experimental and comparative studies.

**Student invented representations, and multiple representations**

These had constructivist roots in that they were used as evidence of students’ active participation and their ability to compare and transform their basic ideas, building more and more abstract ones.

Another shift that accrued due to the constructivist research programme was in the exploration of role of representations in mathematics. At the younger levels, researcher explored the ways in which children would build their own representations of ideas with increasing sophistication. Maher, Speiser, Friel and Konold (1998) demonstrated repeated instances in which children’s reasoning about probability was affected by the ways in which they recorded their results and communicated those with peers and teachers. Likewise, Fuson (1988) explored how children generated algorithms often revealed a spatial orientation on the page that made the possibilities of success more or less likely, and hence how restricting students’ use of format prematurely could frustrate the expression of their competence.

At the more advanced levels of mathematics, the focus was on the use of multiple representations. Instead of assigning the most prestige to the most symbolic of representations, researchers discovered that different representations afforded students differing insights into the mathematical ideas (Artigue, 1992; Confrey & Smith, 1989; Dreyfus, 1993; Janvier, 1987; Kaput, 1987).

An excellent example of the use of multiple representations came with the movement for algebra to emphasize not only the development of symbolic manipulation skills but to act as a means to understand families of functions. Families of functions were characterized in two ways: (1) A particular family (linear, quadratic, exponential, and trigonometric) was related by their individual familial bonds (rates of change, curve shapes, algebraic form) and (2) The families shared certain general traits as demonstrated in transformations (actions on the classes of functions that provided a measure of generalizability across the families). In the area of transformations, students formed deeper generalizations when they were able to make explanations and predictions in multiple representations (graphs, tables, equations) and this process of explaining the impact
of various parameters, often required them to work in dynamic environments across these representations (Borba, 1993; Dreyfus, 1993).

For example, at PME-18, Borba (1994) discussed how students used a given software, Function Probe (Confrey & Maloney, 1991), as they struggled to coordinate of different representations. A model is presented to show how knowledge can be constructed by such coordination. Borba (1995) illustrated how in such a process of coordinating representations, students may transform the software adding features that were not thought of by the designers (Borba & Confrey, 1996). More recently, Borba & Villarreal (2005) and Borba & Scheffer (2004) have extended the notion of multiple representations and demonstrated the need to extend it to coordination across different media, including graphing calculators, computers, sensors, and paper. Moreover, they claim that this coordination has also to be integrated with a very basic activity within humans’ body motion.

*Socio-constructivist norms*

These were produced as researchers took constructivism into the classroom. As stated in the principles, in order to participate successfully in a constructivist environment, classrooms must shift from a passive to an active role. Some explored how these shifts disrupted normal assumptions under the “didactical contract” (Brousseau, 1984; Chevallard, 1988), and discussed the need for changing the expectations of the students. In 1986, Cobb, Yackel and Wood found that they needed to shift the behaviors of the students as early as first to third grades, to encourage them to listen to other students and to talk about their solutions (diSessa & Cobb, 2004). Drawing on the work of Bauersfeld (1998) and Voigt (1985) and symbolic interactionism, Cobb and his colleagues established that if a teacher were to successfully develop a constructivist orientation among students, s/he would need to “renegotiate classroom social norms.” For example, they explain that constructivist classrooms tend to count as “different” solutions that while producing the same result, represent different cognitive processes. This shift is what is seen as different, changes what is learned in two ways: (1) Different ideas are foregrounded, and (2) Students’ reflections on their own thinking are strengthened. Finally, teacher learning from students is often reported. Other norms explored included what is a clear, acceptable, or sophisticated explanation. In diSessa and Cobb (2004), this area of research was considered to be an “ontological innovation” and led to the identification of this approach as a distinct branch of constructivism, socio-constructivism. Wood, Cobb and Yackel (1995) wrote, “It is useful to see mathematics as both cognitive activity constrained by social and cultural processes, and as a social and cultural phenomenon that is constituted by a community of actively cognizing individuals” (p. 402).
New Topics

The introduction of the technological learning environments in various topics in school mathematics curriculum, such as geometry and statistics, provided new insights into how students learn these topics and how we teach them. For example, dynamic geometry computer environments, such as the Geometer’s SketchPad [GSP] (Jackiw, 1991), Cabri-géomètre (Baulac et al., 1988), the Geometric Supposer (Schwartz & Yerushalmy, 1985), and 3-D dynamic images (Gutiérrez & Jaime, 1993) provided students with different tools for exploring and understanding geometric concepts, and thus offered alternative ways to learning geometry, reasoning about geometry, and constructing proofs (Arzarello, Micheletti, Olivero, Robutti, Paola & Gallino, 1998; Gutiérrez, 1995; Hollebrands, 2002; Laborde, 1993; Marrades & Gutiérrez, 2000). The constructivist perspective of Piaget and the theory of Van Hiele on geometric thinking and proofs are key to the theoretical foundation for use of such computer environments in learning geometry (Clements & Battista, 1992). Furthermore, there is a key issue in considering how mathematics must be reframed when placed into electronic media such as when Balacheff introduced the ideas of computational transposition (Balacheff, 1993) in which he recognized the need in technology for the coordination of mathematical screen representation with the underlying computational models.

Furthermore, using statistics as an example, one can see how constructivism has spawned inventive and original ideas with mathematical import as well as identifying key landmarks in learning. As statistics and data analysis are becoming focal areas of mainstream school curricula in many countries, attention on research in statistics education is paid to the students’ development in statistical thinking and reasoning, students’ understanding of statistical topics, and the use of dynamic statistical computer software, such as Fathom (Finzer, 2001), TinkerPlots (Konold & Miller, 2005), and Minitools (Cobb, Gravemeijer, Bower & McClain, 2001). Although mathematics provides the theoretical foundations for statistical procedures, statistics education does not necessarily conform the traditional approaches to teaching and learning mathematics. For instance, statistics is a relatively new curricular area in which the new content and approaches to data analysis in statistics education develop as the field of statistics changes with the new techniques of data exploration and data analysis tools (Ben-Zvi & Garfield, 2004; Biehler, 2001). Moreover, the use of technological tools clearly offers new ways to support the development of students’ statistical reasoning through providing students with dynamic construction of statistical concepts (Bakker & Gravemeijer, 2004; Ben-Zvi, 2000). Furthermore, in the field of statistics education research, one can trace the shift in emphasis on statistical topics and ideas in instruction: From a focus solely on measures of central tendency to the idea of variation in reasoning about data (Shaughnessy, 2004); focus on the big ideas, such as the notion of distribution, rather than a collection of loosely related topics (McClain, Cobb & Gravemeijer, 2000); and linking probability and statistics through the topics of variation on probability sample space (Reading &
A 30-YEAR REFLECTION ON CONSTRUCTIVISM

Shaughnessy, 2000) and statistical inference (Pfannkuch, Budgett, Parsonage & Horring, 2004; Watson & Moritz, 1999).

Assessment

One major effect of constructivism was that it opened up the topic of assessment. Assessment was viewed as a means to support constructivist practices in a variety of ways. First, concerns were raised that the traditional testing approaches failed to evaluate students’ knowledge sufficiently with their focus on multiple-choice format or solely on the production of answers. Secondly, assessments were viewed as key contributors to students’ awareness of their own learning and to increasing their ability towards reflective abstraction (Bell, Swan, Onslow, Pratt, Purdy, et al., 1985; Simon, Tzur, Heinz & Kinzel, 2004). Thirdly, researchers focused on using richer tasks to give teachers increased understanding of student reasoning, and as a means to support constructivist curricular changes, and to strengthen teachers’ diagnostic teaching (Schoenfeld, 1998). The Shell Centre in Nottingham linked up to researchers in the United States (Berkeley, Michigan State) to form MARS as web-based resource for the development of these initiatives (http://www.nottingham.ac.uk/education/MARS) and Balanced Assessment (http://www.nottingham.ac.uk/education/MARS/services/ba.htm). Given the importance of assessment in instruction, a number of researchers have realized that providing teachers direct access to artifacts of student work proves to be an excellent means to engage them in examining their own beliefs and in looking more deeply into student thinking and reasoning.

Teaching and teacher education

Constructivist theory has had a dramatic effect on teacher education. It has been repeatedly debated whether constructivism entails a theory of teaching (Bauersfeld, 1995; Kilpatrick, 1987; Simon, 1995; Steffe & Gale, 1995). The argument is a critical and complex one. It derives from the question “what is constructivism a theory of?” If it is an explanatory model of how learning occurs, then it is not clear that it can be directly transformed into a normative theory of what teachers ought to do? If it is a theory of “good learning”, then what a teacher should do is to promote constructivist learning. Or possibly, constructivism itself is not sufficient to produce a theory of teaching; an additional theoretical framework for mathematics pedagogy are needed (Simon, 1995, p. 117).

To resolve this dilemma, a number of researchers have been engaged in developing pedagogical frameworks that focus on the design of tasks, planning of lessons, stimulating, guiding and supporting students’ discourse and activities, creating a learning environment and analyzing and assessing student work and progress. Brousseau’s work on the didactical contract and how to devise tasks that leads students to take responsibility for the problem, devolution is one such example (Balacheff, 1990). Douady (1986) discussed how this becomes “situations for institutionalization”. Confrey (1998) discussed the dialectic of voice and
perspective, where the voice of the student is interpreted through the teachers’ perspective and likewise, the perspective of the teacher is transformed as she views her on knowledge through the voice of the student. Similarly, Ball (1993) refers to the “bifocal perspective” (p. 159). Ball described teaching as “essentially an ongoing inquiry into content and learners and into ways that contexts can be structured to facilitate the development of learner’s understanding” (p. 166). And researchers examine how teachers negotiate student trajectories of learning (Confrey, 1998; Simon, 1995).

These initial work on teaching has been complemented by the profound influence of constructivism on teacher education around the world. Much of this has been stimulated through an ongoing series of working groups by PME members and the products of scholarship produced by them. They recognize that teachers need to both learn about constructivist learning, and experience mathematics from a constructivist perspective. Researchers in PME have thus conducted studies of teacher education (Ball, 1993; Bauersfeld, 1995; Jaworski, 1991, 1994; Ma, 1999; Simon, 1988) and at least three volumes of studies (Jaworski, Wood & Dawson, 1999; Ellerton, 1999; Zack, Mousley & Breen, 1997). In most cases, they have determined that teachers need time to both engage with the material as learners within a constructivist paradigm and to consider what this implies for their practice. At the current time, there is particular interest in three arenas: How to describe the nature of teachers’ knowledge as illustrated in Ma’s (1999) “profound understanding of fundamental mathematics”, in Ball’s current characterizations of teacher knowledge (Bass & Ball, 2005), and in Japanese professional development process “Lesson Study” (Yoshida, 1999).

Methodology

Over the course of the past thirty years, there have been a number of developments in the methods of conducting research. Piaget was the inventor of the “clinical method” (Piaget, 1976), which led to extensive use of clinical interviews by scholars in PME. Writings on the conduct of the interview were presented by Ginsburg (1997), Opper (1977) and Steffe, Cobb & Von Glasersfeld (1988). One major shift in that work was the recognition that the purpose of the interview is for the interviewer to build a model of the student’s evolving conceptions. The role of the interviewer became clearer, the interviews extended over longer periods of time, and interviews were described by learners as learning experiences, the clinical interview began to be replaced by teaching experiments (Cobb, 2000; Confrey & Lachance, 2000; Lesh & Kelly, 2000; Simon, 2000). These were repurposed from the Russian didactical community and led to extended studies of student and student-teacher interactions. Simultaneously, the learning sciences evolved as a sub-field (Sawyer, in press) and as it did, design experiments were added to the repertoire of constructivist methodologies. In “Design Experiments in Educational Research”, Cobb, Confrey, diSessa, Lehrer and Schauble (2003) identified five cross-cutting features of design experimentation towards “develop[ing] a class of theories about both the process of learning and the means
that are designed to support that learning, be it the learning of individual students, of a classroom community, of a professional teaching community, or of a school or school district viewed as an organization” (p. 10). These five features of design research indicated that they were: (1) Highly interventionist in vision, (2) Design-based, (3) Theory generative, (4) Built and revised based on iterative revisions and feedback, (5) Ecologically valid and practice oriented. More recently, Confrey (in press) wrote a chapter on the evolution of design studies and in it, she described the goal of design studies to provide a perspective on conceptual corridors in which one creates a set of constraints for development within a broad web of relations and then conducts design experiments to identify likely landmarks and obstacles along that corridor. The expectation is not to reproduce a learning trajectory to be increase the likelihood of successful learning by defining a corridor of possible opportunities and constraints and recognizable typical patterns for use by teachers in conducting practice.

In reviewing these nine areas of constructivist practice (bridging theories, grounding in activity and tools, alternative perspectives and developmental sequences, multiple representations, socio-constructivist norms, new content topics, assessment, teaching and teacher education and methodology), we have sought to acknowledge that constructivism has had a profound impact on the field. Further, we have identified a set of key themes that provide critical links between the ten principles and the ongoing practices of schools.

PART 5: IS CONSTRUCTIVISM PROGRESSING, DEGENERATING, OR STATIC?

Constructivism’s influence on mathematics education is, in our opinion, unfortunately, waning while simultaneously being welcomed anew in places where it has not been yet, such as Singapore and Turkey. One wonders if this is because some of its key ideas have been “taken for granted” so completely they bear little explicit reference or constructivism’s potential was exhausted, other ideas are proving more compelling, or because the field tends to be fickle, changing theories too frequently. We suspect that two influences account for the shift: (1) The field lacks maturity in how to use theory and systematically accumulate findings and results; and (2) The understanding of the social-cultural, political and economic forces that influence mathematics learning has drawn attention away. While we acknowledge the value of situating constructivism in relation to broader patterns and trends, we warn that attention to issues of content must be maintained if the work is to adequately serve mathematics education. Even as we offer these observations, we see areas in which progress continues and thus we end the chapter with a discussion of these topics as particularly worthy projections of the constructivist research programme.

We suggest that constructivism has not been adequately understood as a grand theory. Drawing on Lakatos (1976), we would argue that constructivism constitutes a research programme, with a theoretical hardcore, that does not directly predict empirical data, but is protected by a belt of bridging theories. At its core, we are arguing that the ten principles outlined in the second section provide a description
of the theoretical hardcore of constructivist theory and avoid the polarization into trivial and radical constructivism generated by the two-principle summary. We see the epistemological dimension of the framework, as captured in genetic epistemology as central to the theory and as holding further potential for investigations. Thus, we reject statements which suggest that constructivism is solely individualistic, solipsistic, or lacking a view of objectivity (Goldin, 1989, 2002).

In order to address our concerns about the use of theory to inform practice and the means to evaluate the quality of empirical data, we argue for the need for more specific and elaborated bridging theories. These theories do more than to restate the principles, as they should spell out how the theory translates into practice and yield testable and refutable hypotheses and conjectures. A challenge presented by this approach is that bridging theories are often curricular specific, and the generalizability of the research for international consumption may become restricted and difficult to share at that level. This may force us to articulate new ways of conducting and reporting research. We see three kinds of theories that provide the kinds of theory-driven empirical work we have in mind.

One set of bridging theories has an explicit design or engineering orientation and presents an overall theory of instructional approach. The design orientation embraces the novelty associated with design propelling it beyond what has been established in research and yet addressing all the elements of instructional practice, in an ongoing way through design studies or teaching experiments conducted over long periods of time. The goal is document the effects of the theoretical work on students’ reasoning across a variety of topics, and should lead to clear forms of prediction based on the continuity and duration of the studies. Instructional design theories, such as RME, or didactical engineering, represent such examples.

Curricular innovations with subsequent evaluation represent another type of bridging theory (Mathematical Sciences Education Board, 2004). In these, a full curriculum is developed meeting the needs of the particular context in which it is implemented (e.g., Singapore Math, Connected Math, etc.). Studies on the effectiveness of these curricula report on the intended, enacted and achieved curricula, and link these with a variety of measures of student performance.

A third approach is to articulate clear and explicit views of the structure of particular sub-domain, such as statistics, rational number, or class of algebraic functions and to treat them as a conceptual field (Vergnaud, 1996). In these studies, the emphasis is on how students’ thinking evolves as they are provided sequences of tasks designed to highlight what is possible and likely in students’ approaches and in hypothesizing about their conceptual structures. In all such initiatives, we would expect to see explicit attention to the concepts, situations, properties, proof forms, representations, algorithms and structures. Some kind of genetic-epistemological analysis would accompany the articulation of these constructs of mathematics over time. The empirical work could be comparative, or it could be a documentation what was learned, how and when over what particular students. As stated in Confrey and Lachance (2000), the development of a conjecture followed by clear, measurement-based indicators of student work (i.e., test results, scoring
by rubric, and other means to dimensionalizing and categorizing student accomplishments) and thinking would have to accompany the research analysis. The work would be expected to be followed by the development of a stable, repeatable and well-defined product or process that could be tested in a more scaled and systematic way and would depend on the creation of shared outcome measures for use across research initiatives focused on similar or related content. While the product of the design experiments can support multiple realizations and elaborations as stated in the corridor concept (Confrey, in press), key indicators would be held stable to permit comparative analysis within the class over time.

It is easy to become distracted from the central effort of articulating, developing, refining and evaluating conceptual corridors for all of the major strands of mathematics, and to date, attention to some has overwhelmed attention to the other key topics. In reviewing the past thirty years, we see significant amounts of repetition and rediscovery of ideas, a proliferation of new terms for similar occurrences, an absence of negotiation and agreement on controversies, and a paucity of careful empirical study beyond initial experimentation. We further see a dearth of synthesis of prior work with identification of key topics for extensions. We find ourselves guilty of these same sins, and we propose that part of the reason for it is that too often, we have worked with a grand theory at a cost to the specificity of a bridging theory. Bridging theories are more likely to be subject to falsifiability if they fail to achieve improved student performance on valid assessments. The field needs to work to reduce its advocacy and strengthen its reliance on evidence that can be understood and accepted by experienced practitioners. This is in part due to the small number of us in the field, and the multiple demands on our time (research, teaching teachers, conducting in-service, preparing materials, etc.), but it has contributed to our tendency to be cavalier about our uses of theory, dismissive of others, and imprecise about our own. Perhaps some of the reason that theories have such short lives is due to this lack of deeper analysis and negotiation of resolutions to controversies. The maturity of the field will be evident when these changes decelerate.

We further believe that socio-cultural perspectives have gained in stature in the field which has diverted attention to constructivist research. While we believe that strengthening students’ agency, their beliefs in their rights, claims and abilities to learn mathematics is a critical element of change, making the subject compelling, organizing for successful and deep learning, and ensuring that teachers know the content in a substantive way are also imperative. We raise the concern that as socio-cultural perspectives have gained force, less and less attention is being paid to the environmental/contextual/physical issues raised by constructivist scholars. As a result, less emphasis is being paid to the mathematics itself, how it is learned and how to communicate that to teachers. We reject the view that this can be cast as simply individualistic vs. cultural or social; we further reject the view that knowledge in the two views should be cast as cognitive vs. culturally distributed. Rather, we prefer the view that mathematics learning entails critical elements of grounded activity and socio-cultural communication and that these components interact in important and interesting ways. Constructivism, more than any other
JERE CONFREY AND SIBEL KAZAK

theory to date, has emphasized the importance of development and growth, in understanding learning. It clearly recognized that both biological-physical-environmental forces and social-cultural-political forces affect that process. What constructivism did is to locate the primary source of mathematical knowledge in patterns that can be generated in relation to biological/physical/environmental surroundings, and to recognize how the socio-cultural context signifies, facilitates/retards, and shapes that learning, while keeping the individual as the primary unit of analysis embedded in groups, classes, schools, communities and cultures. Neither influence is viewed as primary, nor can either instance be in fact separated due to our membership as observers and participants in all these enterprises, at all times.

A revised grand theory that draws upon constructivism and socio-cultural perspectives fully enough to satisfy the proponents of each is likely to emerge. This theory will need to address both theoretical and methodological elements and is likely to propose a mixed method approach that is staged over time. It will need to address in a broader way than constructivism how to engage students in the reasons for the pursuit of mathematical or scientific proficiency, and pay careful attention to the larger social and cultural issues surrounding such decisions. In addition, it will recognize the acculturation involved in engaging in the practice of mathematizing, while ensuring careful attention to the development of independent thought and precise patterns of reasoning. To do this, multiple units of analysis will be precisely and carefully linked.

In projecting forward the future development of constructivism, we see the need to continue to recognize the changes in a technological society and their impact on schooling. Research on how professionals and experts work effectively (Greeno & Hall, 1997; Hall, 1995, 1998) provide insight into the real ways in which the socio-cultural and the environmental/physical/tool-based, media-based business of technological work is conducted, and too few students in schools have knowledge of how mathematics is embedded in these diverse settings, much less what kinds of work are available and needed. These environments/working circumstances have potential to both attract students into quantitative disciplines and compel them to work hard to be successful. What then remains is to consider how such connections are linked to the constructivist research programme.

We predict, with others, that research on modeling in mathematics represents a key bridge and is likely to be one rightful successor to constructivism (Confrey & Maloney, 2005; Gravemeijer & Stephan, 2002; Lehrer & Pritchard, 2002; Lesh & Doerr, 2003). Within such a perspective, one views knowledge construction as the mapping of and exploration of the systematicity of relations between a base and a target domain. The reason we make a claim of succession is epistemological, as modeling can address the dual challenges of providing a focus on coherence views of truth, while replacing the correspondence theory with a viable alternative. Correspondence is cast as the relationship between two domains of understanding, one secure and then other more uncertain, rather than between an individual and an external reality. However, those two domains can be of varied levels of abstraction. Moreover, one can compare and contrast a variety of types of models, and hence
produce a more nuanced continuum for guiding students’ mathematical development. Lehrer and Schauble (2000) provided a four-part taxonomy of modeling designed to move from literal resemblance to relational structure: Physical microcosms (e.g. a physical model of an elbow), representational systems (e.g. a map), syntactical models (e.g. modeling phenomena as a coin toss) and hypothetical-deductive models (e.g. modeling gas as collisions of billiard balls). As students move increasingly towards the hypothetical-deductive models, one expects significant use of coherence as a means of deciphering relationships and producing predictions beyond the physical correspondences, obtaining the mathematics of axiomatic systems. At the other end of the spectrum, children are confronted with the challenge of understanding how one explains the source of mathematical ideas can be rooted physical settings. Representational systems begin to reveal the potential insights obtained by comparing and contrasting multiple representations, in which consistency is sought, while differences are used to highlight new features and assist in establishing warrant for various conjectures.

Across the spectrum, Lehrer and Schauble emphasize the mathematical underpinnings of the effort in the acts of quantification, the creation of measure, the understanding of data and probability, and/or the development of a spatial form of reference. A modeling approach brings together mathematics with other disciplines, while also reserving significant time for developing its internal relations and meanings. Much more work remains on how the development of model-based reasoning emerges, but we see significant promise in this area as the epistemological successor to constructivist epistemology.

Overall, constructivism has had an impressive impact on mathematics education, in that it has propelled the children into the forefront of activity and asked genuine questions about how to make effective use of the resources, language, inscriptions, and ideas they bring to the enterprise of learning. It has produced many practical accomplishments from curricula to new technological tools, as well as documented a number of substantial considerations of student thinking about which all teachers need to know. Because of the theory, we have realized that careful attention must be paid to how students become increasing aware of what they believe and know, and how this is refined and developed in the company of others. Our views of the role of teachers has been transformed to recognize their critical contributions as stimulators, guides, facilitators and critics—assisting students in developing the fundamental reasoning abilities that are the hallmark of mathematics as students complete a tour of the rich variety of topics in the fields. We end our chapter with the humble recognition that the task we undertook, to summarize thirty years of scholarship across the globe, was impossible; and that we have been able only to identify some critical moments, to summarize some key principles, and to reflect on some of the major legacies of constructivism. We have not recognized all the contributions adequately and apologize in advance to those whose ideas were not selected to illustrate the key points. Our examples were no doubt influenced by our own context and experience of the field, though we work diligently to recognize contributions across the world. With these limitations in mind, we can say without a doubt, that PME has played a
most substantial role in the development of constructivist theory, critical in making
the theory an international effort. It is our hope, then, that the synthesis offered
here can assist in guiding us to continued productivity and significant and
compelling advances over the next thirty years.

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REFERENCES

Herscovics, & C. Kieran (Eds.), Proceedings of the 11th PME International Conference, 3, 236–
242.
Graham (Eds.), Proceedings of the 16th PME International Conference, 3, 195–216.
and modalities of transition from conjectures to proofs in geometry. In A. Olivier & K. Newstead
(Eds.), Proceedings of the 22nd PME International Conference, 2, pp. 32–39.
Garfield (Eds.), The challenge of developing statistical literacy, reasoning and thinking (pp. 147–
Learning from computers: Mathematics education and technology (pp. 131–158). Berlin, Germany:
Springer.
fractions. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), Rational numbers: An
integration of research (pp. 157–196). Hillsdale, NJ, USA: Lawrence Erlbaum.
Hirabayashi, N. Nohda, K. Shigematsu, & F.-L. Lin (Eds.), Proceedings of the 17th PME
International Conference, 2, pp. 97–104.
II Follow-up Conference, Atlanta, GA, USA.
a social practice. In L. P. Steffe & G. Gale (Eds.), Constructivism in education (pp. 137–158).
Hillsdale, NJ, USA: Lawrence Erlbaum.
& J. Garrison (Eds.), Constructivism and education (pp. 195–212). New York, NY, USA:
Cambridge University Press.
France: Cedic-Nathan.
A 30-YEAR REFLECTION ON CONSTRUCTIVISM


Bell, A., Swan, M., Onslow, B., Pratt, K., Purdy, D., et al. (1985). Diagnostic teaching for long term learning (ESRC Project No. HR8491/1). Nottingham, UK: Shell Centre for Mathematical Education, University of Nottingham.


JERE CONFREY AND SIBEL KAZAK


A 30-YEAR REFLECTION ON CONSTRUCTIVISM


JERE CONFREY AND SIBEL KAZAK


A 30-YEAR REFLECTION ON CONSTRUCTIVISM


Nesher, P. (1988a). Multiplicative school word problems: Theoretical approaches and empirical findings. In M. Behr & J. Hiebert (Eds.), *Number concepts and operations in the middle grades* (pp. 19–40). Reston, VA, USA: Lawrence Erlbaum and NCTM.
A 30-YEAR REFLECTION ON CONSTRUCTIVISM


343
JERE CONFREY AND SIBEL KAZAK


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